THE

MATHEMATICAL GAZETTE.

LONDON:
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AND BOMBAY.

JOHN BOLYAI'S "SCIENCE ABSOLUTE OF SPACE."

I.

ALTHOUGH some years have elapsed since the first appearance of Professor Halsted's translations of Lobatchewsky's *Theory of Parallels* and John Bolyai's *Science Absolute of Space*, they are not so widely known and studied as they deserve to be. There is nothing in the mathematics of these small volumes of a kind to deter any one from reading them; they are short and elementary treatises, disclosing a system of geometry quite distinct from Euclid, by rejecting the axiom concerning parallels. The *Science Absolute of Space* is a most interesting exposition of one branch of non-Euclidean geometry; but it is unfortunately extremely condensed, and occasionally obscure. It would have been a great advantage if explanatory notes had been added to the English translation.

Lobatchewsky and Bolyai both admitted the whole of Euclid's assumptions with the exception of the 12th axiom; in place of which, they supposed that if a perpendicular AB were drawn to a given straight line BP, and the point P were taken on BP at an indefinitely great distance from B, the angle PAB would converge to a definite value, depending on the distance AB, and less than a right angle. They investigated, to the fullest extent, the consequences of this supposition; knowing that if they should prove that the rejection of the axiom led to any inconsistency, they would thereby, to a great extent, remove all existing doubt as to its truth; but if not, they would be laying the foundation of a new geometry. In this latter respect they succeeded probably far beyond their anticipations. Starting with the same assumptions, they arrived at identical results; but while Lobatchewsky deliberately gives only a part of what he had worked out, the geometry of Bolyai is elaborate and complete.

Before proceeding to describe the *Science Absolute of Space*, we shall try to explain, in what appears to be the simplest way, the relation of that particular variety of space in which the geometry of Bolyai holds true to three-dimensional space in

general. We confine ourselves to one method of explanation; a method which is simple and elementary, although, from the nature of the case, tentative and very possibly capable of further generalization. This method is based on the supposition that space possesses of itself certain properties and *limits* of extension, and that lines, surfaces, and solids may occupy or fill up certain portions of it. By the *limits* of extension, it is not implied that the space considered does not extend to infinity, but only that the directions in which it extends from any point are limited. We may refer the reader, who is interested in the general subject, to Chrystal's *Non-Euclidean Geometry*, reprinted in pamphlet form from Vol. X. of the "Proceedings" of the Royal Society of Edinburgh, and to Halsted's translation of Vasiliev's address on Lobatchewsky at the University of Kasan in 1893. The numerous modern developments of the subject have been chiefly on the lines marked out by Riemann, Helmhotz, Clifford, Cayley, and Klein.

The space of the visible universe, from which the first notions of space are derived, is three-dimensional—*i.e.* it extends from any point in three mutually perpendicular directions, and in all directions dependent on these three both forwards and backwards, but in no other directions. Since we have no knowledge of any direction at a given point which does not belong to our space, we are apt to assume that there are no directions in which it does not extend. In order, however, to arrive at a more comprehensive view of space in general, it is very desirable to free our minds from this assumption; and to suppose, on the contrary, that at every point in space, as we know it, there are lines extending directly outside that space, of which one at least is perpendicular to every line in it through the given point.

There is another way of expressing the same idea which may make it more easy to understand. It is fully admitted that Euclidean geometry is founded on perfectly legitimate assumptions, and that there is a conceivable space in which all the results established in that geometry are true; but it does not follow that they are more than approximately true in our space; or, in other words, it does not follow that ours is Euclidean space. Euclidean geometry deals with straight lines, regarding which certain properties are assumed; and one of the most important features of this geometry is that it supplies us with the notion of absolute parallelism, or the notion of direction independent of position. The following question then has to be considered: Does our space extend in precisely the same directions (in the Euclidean sense) from one given point as it does from another? It cannot be proved with absolute certainty that it does; the only evidence that an affirmative answer is correct is that the tests which have been applied have not so far established the

negative. All that can be said is that if it does, it is Euclidean space; but if it does not, it is not Euclidean space, and there must then be directions at any point in which it does not extend, while a Euclidean straight line cannot in general be drawn in it connecting two given points, and the sum of the three angles of a triangle will not be in general equal to two right angles. This last property supplies a test as to whether our space is Euclidean or not; the sum of the angles of the greatest triangles available for measurement has not been shown to differ from two right angles by a greater amount than would naturally result from the errors of observation.

The conception of space, as we have tried to explain it, is that it does not necessarily extend in the same directions from one point as from another, just as a curved surface, or a two-dimensional space, does not. If we take this view of space, it is easy to see that the great principle assumed in Euclid is that made use of in the method of proof by superposition; but in addition to this, there is scarcely a theorem in Euclid the proof of which does not depend on the further assumption that two straight lines cannot enclose a space. The assumption made in using the method of superposition is that space is uniform throughout, in the same sense that a plane or a sphere is uniform, so that a rigid body in one part of space is free to move to any other part, and *fits into* the space in the new position without any change of shape; while so long as a body has not three points kept fixed, it is still free to move in some way. In the general three-dimensional space, the shape of a moving body would continuously change if it continued to fit into the space as it moved. There can scarcely be a doubt that our space is uniform, since the motion of the earth produces no measurable change of shape. In what follows we shall therefore only consider uniform space, which includes Euclidean space as a particular case; and we shall find that, by the method of superposition, many properties of uniform space can be proved. It is, however, necessary to observe that reliance must not be placed on deductions made by analogy from the properties of Euclidean space.

Euclid defines the straight line between two points as the shortest line connecting them; but for our purpose this definition must give place to another, viz.:—The geodesic between two given points in a given three-dimensional space is the shortest line between them which lies in that space. In Euclidean space geodesics are straight lines, and in other uniform space geodesics may be said to be relatively straight; but it will be clearer, for the present, to restrict the term straight to Euclidean straight lines, until we have shown in what respects geodesics in general may differ from them. The important

points of differences are that, whereas straight lines are assumed to be unclosed and infinite, geodesics may be closed and finite, and all geodesics through any fixed point A may intersect again in another fixed antipodal point A'. Geodesics are in fact curved as compared with straight lines, though their curvature may be undiscoverable. Through two given points A, B, which are not a pair of antipodal points, one and only one geodesic can be drawn; and a geodesic is determined without exception when a point A on it is given and the direction in which it starts from A.

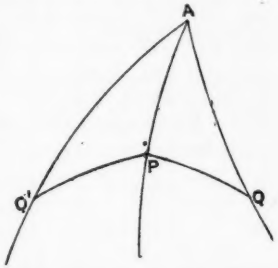
If a geodesic be moved about in any manner without change of shape it remains a geodesic in all its positions. For if a geodesic ABC is moved to any new position $A'B'C'$, and $A'B'C'$ is not a geodesic, then there is a shorter path from A' to C' than $A'B'C'$; and by supposing $A'B'C'$ to be moved back to its original position ABC, it follows that there will be a shorter path from A to C than ABC, which is impossible. Thus any arc of a geodesic can be superposed on the same geodesic at any part of it, so that a geodesic is a uniform line. Also any arc of a geodesic, after being turned through two right angles, can be made to coincide with itself, back to back. In this respect geodesics differ from other uniform lines in the same space, in the same way that a straight line differs from a circle. If geodesics are closed lines they are all of the same length, since any one can be made to coincide with any other. The rays of light from a sufficiently brilliant star might be visible along the whole length of a closed geodesic, and in that case the star could be seen from any point in space in two opposite directions.

A rigid body can rotate about any geodesic AB. Taking AB to denote the geodesic AB, as in the case of a straight line, let C be any point of the body not lying on AB, or AB produced; suppose CN drawn perpendicular to AB, and let C' be any point on the circle described with N as centre, and geodesic radius perpendicular to AB and equal to NC. Then NC can be superposed on NC', while ANB retains its position unchanged; and the triangle CAB will then coincide with $C'AB$. Hence if C' be supposed to move continuously round its circle, and CAB to be brought successively into each of the positions of $C'AB$, it is clear that the rigid body will rotate round AB. Also, if A, B are kept fixed, the rigid body can only move by rotation round AB, for the shortest line in the body connecting A, B must then remain fixed. There is an exception

when A, B are a pair of antipodal points, the body having then the same degree of freedom of motion as if only one point in it were kept fixed.

If two geodesics intersect perpendicularly at a point A, and one of them, produced indefinitely in both directions, be rotated about the other through two right angles, it will generate a *geodesic surface*. Geodesic surfaces correspond to Euclidean planes, and have many similar properties; the chief difference being that if the geodesics are closed lines, so also must the geodesic surfaces be closed. A closed geodesic is a circle, and a closed geodesic surface a sphere, though not a Euclidean circle or sphere, because the space in which they exist is not Euclidean.

A geodesic surface answers to Euclid's definition of a plane in respect to the property that the geodesic joining any two points on it lies wholly within the surface. For let P, Q be any two points on the surface; then, by the definition, the geodesics AP, AQ lie on the surface. Draw a third geodesic AQ' on the surface so that the angles PAQ, PAQ' are equal, and also AQ, AQ'. It is clear that if the geodesics PQ, PQ' do not lie on the surface, they must leave it at P on the same side, whether they touch it at P or not. If then APQ be rotated round AP until AQ coincides with AQ', the geodesics PQ, PQ' must coincide; but they will now leave the surface at P on opposite sides, which is impossible. Therefore PQ must lie on the surface; and it follows that a geodesic surface is uniform, any part of it being superposable on any other part, and also that if any portion be rotated round a geodesic on the surface through two right angles, it will again coincide with the surface back to back.



Two geodesic surfaces which touch must coincide altogether. A geodesic surface is completely determined by three given points which do not lie on a geodesic. If two geodesic surfaces intersect they must intersect along a geodesic, and can have no other point of intersection. For if two geodesic surfaces intersect at A, they must intersect along a line through A, since the space in which they lie only extends from A in three independent directions; and if B be any other point on this line, the geodesic AB lies on both surfaces, and must therefore be their line of intersection.

By considering a system of geodesics drawn on the same geodesic surface, all of which are perpendicular to a fixed geodesic on the surface, a distinction may be made between three kinds of uniform space, viz., elliptic or finite space, in

which the sum of the angles of any triangle is greater than two right angles; Euclidean space, in which the sum of the angles of any triangle is equal to two right angles; and hyperbolic or infinite space, in which the sum of the angles of any triangle is less than two right angles. Elliptic space is defined by the property that any two geodesics of the system considered approach one another, and ultimately intersect. Let AC, BD be two geodesics perpendicular to the fixed geodesic AB; and let AC, BD be equal, and on the same side of AB. Then, in elliptic space, since AC, BD approach one another, we have CD less than AB. Also AC, BD approach one another at a continually increasing rate the further they are produced; *i.e.* if EF bisects AC and BD, then $EF-CD$ is greater than $AB-EF$. To prove this, draw AA', BB', CC', DD' perpendicular to EF. Then AB is greater than EF, and less than $A'B'$; hence A', B' lie on EF produced, and C', D' between E and F. Also the four triangles AEA', BFB', CEC', DFD' are equal in all respects, as will be seen later; hence we have

$$AB < A'B' < EF + A'C' \quad \text{or} \quad AB - EF < A'C',$$

and

$$CD < C'D' < EF - A'C' \quad \text{or} \quad EF - CD > A'C',$$

therefore $EF-CD > AB-EF$. It is to be observed that the reasoning holds no matter how small AE, BF may be; so that if AC, BD begin to approach, they will continue to approach, and at a continually increasing rate, till they meet. Thus it follows that if AC, BD approach one another at all, they must ultimately intersect; so that a uniform space in which two such geodesics as AC, BD approach, but do not ultimately meet, seems to be impossible.

Let AC, BD be produced to meet in K, then it follows from above that $KA = KB$; and if KA, KB be produced on the other side of AB, they will meet in a point K' at the same distance as K from AB. Hence there are two cases to be considered according as K, K' are the same point or not. If K, K' coincide, then KAK', KBK' are closed lines, and all the geodesic are closed and equal. In such a space, although there is no exception to the property that two points determine one and only one geodesic, the fourth proposition of Euclid is not true without limitations; for when the extremities of the bases of two triangles coincide the bases might not themselves coincide, but form together a closed geodesic. Supposing, on the other hand, that the points

K, K' do not coincide, it can be proved that every geodesic through K passes also through K'; for if KBK' be rotated about KAK' every new position of KBK' gives a geodesic through K, K'; and since any one of these might be rotated about any other, and so on, it follows that a geodesic starting from K in any direction will pass through K'. Also any geodesic from K to K' must be half a complete geodesic; for if KBK' be rotated about a perpendicular geodesic KK' through two right angles, the initial and final positions of KBK' together form a closed geodesic. This type of elliptic space, in which to every point K corresponds an antipodal point K', is sometimes called spherical space. A curious property of this space is that every circle and sphere has two antipodal centres. In both simple elliptic and double elliptic or spherical space the geodesics are finite closed lines; and Euclid's axiom, that two straight lines cannot enclose a space, does not hold. Propositions 5, 6 in the first book of Euclid are, however, true in elliptic space; and Propositions 4, 8, 26 are true for triangles whose sides are less than half the complete length of a geodesic.

Conversely it would seem that uniform space in which the geodesics are closed must be elliptic space. For since two geodesics KA, KB through K intersect again, let K' be the first point in which they meet after leaving K. Then, whether K' coincides with K or not, the geodesic AB joining the middle points of KAK', KBK', will lie on the geodesic surface determined by KAK', KBK', and will be perpendicular to KAK', KBK', by symmetry. Hence the geodesics AK, BK, drawn perpendicular to AB, intersect, and therefore belong to elliptic space.

Euclidean space is defined by the property that a system of geodesics on a geodesic surface perpendicular to a fixed geodesic neither approach nor recede from one another. From this it can be easily proved that any geodesic on the surface which is not perpendicular to the fixed one will intersect all the others; and it would appear to follow from above that the geodesics cannot be closed, since they do not belong to elliptic space.

Hyperbolic space is defined by the property that the system of geodesics perpendicular to a fixed geodesic recede from one another; and it can be proved, as above, that they recede at a continually increasing rate the further they are produced. In this space a geodesic which is not perpendicular to the fixed one does not intersect all the perpendicular geodesics; and the geodesics are unclosed lines of infinite length. The geometry of Lobatchewsky and J. Bolyai is the geometry of hyperbolic space; we must however reserve, till the next number, a description of the results of Bolyai's remarkable investigation.

EDITOR.

NOTES ON ELEMENTARY DYNAMICS. I.

IN working problems on Collision or Impact, the following method is often useful:

Imagine a rigid plane moving with velocity equal to that of the centre of gravity of two bodies, so that during collision it touches and separates the surfaces of contact. (We may either suppose that the mass of the plane is infinite, or that it is constrained to move as stated.) Then the motions of the two bodies due to their mutual impact will obviously be the same as if each separately had impinged on this plane. Thus the problem of two bodies is reduced to two simpler problems of one body impinging on a body whose motion is fixed. And we may by giving to each of the bodies in question, in addition to its own motion, a motion equal and opposite to that of the centre of gravity of the system, bring the plane of impact to rest, so that the motion of each body relative to this plane is the same as that of a body impinging on a fixed body.

N.B.—In what follows, by *relative velocity* is to be understood that *relative to centre of gravity.*

This method can be applied either in working out the ordinary book-work results or in problems, and often enables us to write down at once, or with slight manipulation, the result required, instead of having to solve simultaneous equations.

Another useful principle is that the kinetic energy of a system is equal to the sum of the K.E. of the total mass moving with the velocity of the centre of gravity and that due to the motions of the parts relative to the centre of gravity. The former is, of course, unaltered by collisions between the different bodies of the system.

EXAMPLE 1.—*Given masses m_1, m_2 with velocities u_1, u_2 in the same direction, to find the velocities v_1, v_2 after impact, the coefficient of resilience being e.*

$$\text{Velocity of centre of gravity} = \frac{m_1u_1+m_2u_2}{m_1+m_2}.$$

$$\text{Relative velocity of } m_1 \text{ before impact} = u_1 - \frac{m_1u_1+m_2u_2}{m_1+m_2} = \frac{m_2(u_1-u_2)}{m_1+m_2},$$

$$\therefore \quad \text{,,} \quad \text{,,} \quad \text{after} \quad \text{,,} \quad = -\frac{em_2(u_1-u_2)}{m_1+m_2},$$

$$\therefore \quad v_1 = \frac{m_1u_1+m_2u_2}{m_1+m_2} - \frac{em_2(u_1-u_2)}{m_1+m_2}.$$

Note that to get velocity after impact we may either use the rule that it is = velocity of centre of gravity $-e\times$(rel. vel. before impact), or that it is = velocity before impact $-(1+e)\times$(rel. vel. before impact), the latter expressing the fact that the relative velocity has been destroyed and e times its value generated in the opposite direction.

Again, change of velocity of m_1 = change of relative velocity of m_1

$$= -(1+e) \times (\text{rel. vel. before}) = -(1+e)\frac{m_2(u_1-u_2)}{m_1+m_2},$$

$$\therefore \quad \text{impulse on } m_1 = -(1+e)m_1m_2\frac{u_1-u_2}{m_1+m_2}.$$

EXAMPLE 2.—*To find loss of energy through impact, the same data being given.*

The energy due to motion of centre of gravity, viz., $\frac{m_1+m_2}{2}\left(\frac{m_1u_1+m_2u_2}{m_1+m_2}\right)^2$ is unaltered.

The velocity of each mass relative to the centre of gravity is diminished in the ratio $1 : e$, and hence the corresponding part of the energy is diminished in the ratio $1 : e^2$. Hence the total loss of energy is

$$\tfrac{1}{2}(1-e^2)\left[m_1\left\{\frac{m_2(u_1-u_2)}{m_1+m_2}\right\}^2+m_2\left\{\frac{m_1(u_2-u_1)}{m_1+m_2}\right\}^2\right]=\frac{(1-e^2)m_1m_2(u_1-u_2)^2}{2(m_1+m_2)}.$$

EXAMPLE 3.—*A ball of mass 2 lbs. impinges directly on a ball of mass 1 lb. which is at rest; find the coefficient of restitution if the velocity with which the larger ball impinges be equal to the velocity of the smaller ball after impact.* (Loney's *Elements of Dynamics*, p. 131, ex. 8.)

Take the velocity in question as the unit.

Then velocity of centre of gravity $=\frac{2}{2+1}=\frac{2}{3}$.

Hence rel. vel. of smaller mass before impact $=-\frac{2}{3}$.

„ velocity „ „ after „ $=(1+e)\times\frac{2}{3}$

(using the second rule for final velocity mentioned above).

Hence $(1+e)\frac{2}{3}=1, \quad \therefore \quad e=\frac{1}{2}$.

EXAMPLE 4.—*A billiard ball A, moving parallel to one side of the table, strikes another ball B (initially at rest) in the centre. After B has struck the cushion and then struck A again, the velocity of A is three-fourths of its initial velocity, but in the opposite direction. Show that if e is the coefficient of elasticity between the balls and also between the ball and the cushion*

$$e^3+e^2+3e=4.$$ WOOLWICH, June, 1894.

For simplicity, let the mass of the ball be chosen as unit mass and its initial velocity as unit velocity.

The velocity of the centre of gravity of A and B is initially $\frac{1}{2}$, and the velocities of A and B relative to this, $\frac{1}{2}$ and $-\frac{1}{2}$. Hence the velocities after impact are $\frac{1}{2}-\frac{e}{2}$, $\frac{1}{2}+\frac{e}{2}$. Hence velocity of B after impact on the cushion $=-e\frac{1+e}{2}$. After this the velocity of the C.G.

$$=\tfrac{1}{2}\left\{\frac{1-e}{2}-\frac{e(1+e)}{2}\right\}=\tfrac{1}{4}(1-2e-e^2),$$

and the rel. vel. of A is

$$\frac{1-e}{2}-\tfrac{1}{4}(1-2e-e^2)=\tfrac{1}{4}(1+e^2).$$

Hence velocity of A after second impact

$$=\frac{1-e}{2}-\frac{(1+e)}{4}(1+e^2).$$

Equating this to $\frac{3}{4}$, we get

$$-3=2-2e-(1+e+e^2+e^3), \quad \therefore \quad e^3+e^2+3e=4.$$

EXAMPLE 5.—*Two buckets, each of mass M, are hung from the ends of a cord passing over a smooth pulley. A mass m is dropped into one bucket from a height h. Find the subsequent motion till the mass m comes to rest relatively to the bucket, the coefficient of resilience being e.*

The motion is the same as if there were *one* bucket of mass $2M$ not acted on by gravity. The centre of mass of the system will therefore descend with an acceleration $(1-\mu)g$, and the relative acceleration of m will be μg, where μ stands for $2M \div (2M+m)$. The imaginary plane of impact descends with the motion of the centre of gravity, and coincides at the first and following impacts with the plane of the bottom of the bucket. Remembering the formulae $v^2=2gh$, $v=gt$, and $h=\frac{1}{2}gt^2$, where t is the half time of flight for a projectile thrown up with velocity v against an acceleration g to reach a height h, we see that the first impact takes place after time $\sqrt{\frac{2h}{g}}$, the velocity of m being $\sqrt{2gh}$, and its rel. vel. $\mu.\sqrt{2gh}$. The rel. vel. of m after first impact is therefore $=e\mu\sqrt{2gh}$, and the relative motion thereafter is the same as if it rebounded from the ground, except that the acceleration is μg instead of g. Thus the initial rel. vel. is diminished each time in the ratio $e:1$.

For the first rebound,

$$\text{the time of flight} = 2 \times \text{vel.} \div \text{acceleration} = 2e\mu\sqrt{2gh} \div \mu g = 2e\sqrt{\frac{2h}{g}}.$$

Hence the total time T from the dropping of m up to relative rest

$$= \sqrt{\frac{2h}{g}} \times (1+2e+2e^2+\ldots) = \sqrt{\frac{2h}{g}}\left(1+\frac{2e}{1-e}\right) = \frac{1+e}{1-e}\sqrt{\frac{2h}{g}}.$$

The relative height of the first rebound $= e^2 . 2gh . \mu^2 \div 2g\mu = e^2h\mu$, and it diminishes at each successive rebound in the ratio $e^2:1$.

Again, total distance moved by the C.G. from the first impact up till rel. rest is $(1-\mu)\sqrt{2gh} . t + \frac{1}{2}(1-\mu)gt^2$, where t is the interval of time, and is

$$= \sqrt{\frac{2h}{g}}\{2e+2e^2+2e^3+\ldots\} = \sqrt{\frac{2h}{g}} \cdot \frac{2e}{1-e}.$$

Hence distance

$$= (1-\mu)\sqrt{2gh}\frac{2e}{1-e}\sqrt{\frac{2h}{g}} + \tfrac{1}{2}(1-\mu)g\frac{2h}{g}\frac{4e^2}{(1-e)^2}$$

$$= (1-\mu) . 2h\frac{2e}{(1-e)^2}(1-e+e) = \frac{m}{2M+m} \cdot \frac{4he}{(1-e)^2}.$$

And this is the distance descended by the bucket up to the point where rel. motion of m ceases. To verify the last result we may calculate thus: The total motion of the C.G. up to rel. rest is $\frac{1}{2}(1-\mu)gT^2$. Thus distance

$$= \tfrac{1}{2}(1-\mu)g\left(\frac{1+e}{1-e}\right)^2\frac{2h}{g} = \frac{(1-\mu)h(1+e)^2}{(1-e)^2}.$$

To get the distance previously calculated we must subtract from this the distance travelled by the C.G. before the first impact took place, viz. $(1-\mu)h$. It will be found that the remainder equals the previous result.

R. F. MUIRHEAD.

WINGATE'S ARITHMETICK.

The 5th Edition by JOHN KERSEY, Teacher of the Mathematicks, at the Sign of the Globe, in Shandois Street in Covent Garden. London 1671.

THIS begins with an "Epistle Dedicatory" by Edm. Wingate and the Preface of John Kersey. The Table of Contents divides the whole book into three parts; one half of the first portion was Wingate's and the other half was Kersey's improvement upon Wingate, while nearly the whole of the second part and all the Appendix are comprised in "chapters that have been entirely composed by the said J. K."

The definition of number has a quaint tag at its end: "Number is that by which everything is numbered or that which answers to the question how many? (unless the question be answered by nothing)." Notation is of interest for the excellent way in which a row of figures is divided into periods of three, each of which periods has a "peculiar denomination or Sirname," such as thousand, million, thousands of millions; but to go further would, in Kersey's opinion, be both troublesome and unnecessary. It would also, he thinks, be both endless and needless to treat of the infinite kinds of different weights and measures used by the different countries, and so he limits himself to a discursive chapter on the English tables with an explanation of the Ounce Troy, references to Acts of Parliament going back to Henry III., the insertion (under the head of Liquid Measure) of "8 gallons—1 Firkin of Ale, Sope, Herring; 1 lb. of wheat, troy weight—1 pint; 10½ gallons—1 Firkin of Salmon or Eels," and of that "ancient verse,"

"Thirty days hath September, April, June, and November,
February hath twenty-eight alone, and each of the rest thirty-one."

The first four rules are treated much in the usual way. An example of subtraction is explained after the following fashion:

93 :
74

4 from 3, can't, so say 4 from 13, 9. Place this down and carry one on to the 7. This makes 8 to be subtracted from 9 and so the final answer comes to 19. The Multiplication Table, commonly called Pythagoras his Table, is given in a chess board form and runs thus up to 9 times 9. To Division is allotted much space. Besides "three remarkable numbers," the dividend, the divisor, the quotient, the reader is furnished with a fourth serviceable term *dividual*, which is employed to signify that portion of the dividend which is pointed off in a long division sum to show how much is actually to undergo the operation of division at any step. To help the memory

"the substance of division by what method soever" is embodied in a verse,

> "Dic quot, multiplica, sub-duc, transferque secandum,

or

> First you must ask, how oft, in Quotient answer make;
> Then multiply, subtract, a new Dividual take."

An illustration of what is actually meant by the remainder is given in the form of a concrete example of distributing £351 among 8 persons. Contracted division, and the proof of division by multiplication, and *vice versa*, lead up to the Rule of Nines. "There is also of Multiplication a *Common Proof* argued from the Multiplicand, the Multiplicator and the Product by casting away Nines, but by that way of proof (though rightly used) a *false* Product will be affirmed to be true: Example, if 3462 be multiplyed by 786, the true product is 2721132, but if I say 4953132 or 3153132 is the Product (or many others which may be given), the proof by nines will confirm them to be true Products, though they be false as will be evident to such as know the *Rule*, which I mention here only to set a brand upon it, that it may be avoided by all lovers of Truth."

Reduction is made easier to the learner by the exhibition of two sets of Tables, from higher to lower, and from lower to higher denominations. "The Rule of Three is so called because by three numbers known or given, it teacheth to find a fourth unknown, it is also called the Golden Rule for the excellency thereof; Lastly it is called the Rule of Proportion for three numbers are propounded, and a fourth proportional unto them is demanded." Under the rule, thus prefaced, come elementary examples of Practice, and also of Interest: and the Rule of Three inverse, and Double Golden Rule are added.

The rest of the rules which deal with whole numbers are the Rule of Fellowship, or Rule of Plural Proportion, and the double rule of the same, Alligation medial and alternate ("whereof alligation sheweth you the mean price of the mistling"), and the Rule of False. Alligation would be inserted in modern books under the head of mixtures: and the other two are different forms of the rule sometimes known as that of Proportional Parts.

"Thus far of Arithmetick in whole numbers only, the doctrine of Fractions ensueth." "Forasmuch as that Prince of Arithmeticians, Diophantus of Alexandria, in divers of his subtil Problems doth mention unity as a number," our author does the same, and conceives a fraction as the result of dividing unity into several equal parts, and taking a certain number of them. The usual rules are given with a few examples. The

most notable point is that no rule is given for dividing by their G.C.M. the numerator and denominator of a single fraction, or for using the Least Common Denominator in the addition of fractions; but the only method of shortening the work is by a sort of insight, whereby a common denominator less than the continual product of all the denominators can be obtained by inspection.

The doctrine of decimal fractions begins as follows:

"It is hard to determine who was the first that brought Decimal Arithmetick to light, though it be a late Invention, but without doubt it hath received much improvement within the compass of a few years by the industry of Artists, and now seems to be arrived at perfection. The excellency thereof is best known to such as can apply it to the practical part of the mathematicks and to the construction of Tables, which depend upon standing or constant proportions, such as Trigonometrical Canons, Tables for computing of Compound Interest, etc., in which cases decimal operations do afford so great help that (in my opinion) many ages have not produced a more useful invention; but it may be objected that decimal arithmetick for the most part gives an imperfect solution to a question, this I grant, yet the answer so given may be as useful as that which is exactly true; for in common affairs, the loss of $\frac{1}{1000}$ part of a grain or of an inch, etc., to wit, any quantity which cannot be seen, is inconsiderable; but I would not be mistaken, for in extolling *Decimals* I do not cry down *Vulgar Fractions*, since experience sheweth that Decimal Fractions are commonly abused by being applyed to all manner of questions about money weight, etc., when indeed many questions may be resolved with much more facility by Vulgar Arithmetick: (Here follows an example.) I might instance the like inconvenience divers ways were it not for loss of time; so that the right use of Decimals depends upon the discretion of the artist."

Here we have Kersey's opinion of the use and abuse of decimals: and a noteworthy opinion it is, for the time at which he lived, and beautiful it is if only for the use of that word *artist*, "at the discretion of the artist." What better phrase could be suggested in connection with that more than ordinary sense of what is fitting for the purpose of a mathematical example, that striving after the right shade and the right perspective which resembles much the similar striving in the kindred branch of art, painting, which has arrogated unto itself the sole right of calling its pupils artists. This very point of the right use of decimals was recently illustrated in the pages of the *Mathematical Gazette*, where formulae were given in which vulgar and decimal fractions were mixed in such a

way as to give the most compendious form. On the question of decimal weights and measures Kersey says that if the greatest integer in any weight were subdivided decimally "the doctrine of Arithmetick would be taught with much more ease and expedition than now it is; but it being improbable that such a reformation will ever be brought to pass, I shall proceed in directing a course to the studious for obtaining the frugal use of such Decimal Fractions as are in his power." The way to change measures of money, of length and of angles from the ordinary to the decimal form is shown, but we see no traces of a definite attempt to formulate a System of decimal coinage. In those days of Charles II. such a cataclysm as the French Revolution was not dreamt of, and decimal coinage was a dream of the "dreamers of dreams."

Examples of Rule of Three and Rule of False, where fractions are introduced, follow; and the well-known example of Hiero's crown is introduced here, and the tale is as told by Vitruvius. "But forasmuch as Vitruvius delivers not the practical operation, I shall here shew the same after the manner of Cardanus, Gemma, Frisius, and other Arithmeticians." The rules for Square Root and Cube Root, with Arithmetical and Geometrical Progressions, bring the book proper to an end. The Appendix which follows is as long as the book.

J. H. HOOKER.

MATHEMATICAL NOTES.

24. *On Dr. Bryan's Paper* (No. 7, p. 8):—"*Annuities treated without Progressions.*"

This is the method commonly used by young actuaries, and the proofs of Mr. Bryan's interesting article are those generally given by candidates at the examinations of the Institute of Actuaries. They are to be found in print in the *Theory of Finance* written by Mr. George King, the able actuary to the Atlas Insurance Company. Mr. King gives also the following, as another alternative proof in addition to the one given by Mr. Bryan, of the fundamental proposition.

To find the present value of an annuity to continue for n years.

Let r be the ratio of the interest to the principal.
Then 1 invested will yield an annuity of r for ever.

$$\therefore \frac{A}{r} \text{ invested will yield a perpetual annuity of } A.$$

$$\therefore \frac{A}{r} = \text{present value of a perpetual annuity of } A.$$

Now, since a perpetual annuity of A commencing from the present
= an annuity of A for n years + a perpetual annuity of A commencing in n years;

$\therefore$ present value of an annuity of A for n years

= present value of a perpetual annuity of A − present value of a perpetual annuity of A to commence in n years

$$= \frac{A}{r} - \text{present value of } \frac{A}{r} \text{ due in } n \text{ years}$$

$$= \frac{A}{r} - \frac{A}{r} \times \frac{1}{R^n} = \frac{A(1-R^{-n})}{r}.$$

PROF. LONEY.

25. *On the condition that the roots of a cubic are real or imaginary.*

Let one real root of the equation

$$x^3+qx+r=0$$

be $2a$; then the other roots may be written in the form $-a+\beta$, $-a-\beta$, where β may be real or a pure imaginary.

Now $$q=-4a^2+a^2-\beta^2=-3a^2-\beta^2,$$

and $$r=2a(a^2-\beta^2),$$

$$\therefore \quad 4q^3+27r^2=-4\beta^2(9a^2-\beta^2)^2.$$

Now $(9a^2-\beta^2)^2$ is positive whether β is real or imaginary, and β^2 is positive or negative as the roots are real or imaginary.

Hence if $4q^3+27r^2$ is positive, the roots are imaginary; if zero, equal; if negative, real.

T. G. VYVYAN.

26. *On Continued Fractions.*

Consider the continued fraction

$$a_1+\frac{1}{a_2+}\ \frac{1}{a_3+}\ \ldots+\frac{1}{a_n}.$$

It is clear that, if p_n/q_n be the n^{th} convergent, both p_n and q_n are of the first degree in a_n; it is also clear that no reduction to lower terms can take place in the course of the work.

Hence $$\frac{p_n}{q_n}=\frac{Aa_n+B}{Ca_n+D} \qquad \text{.....(i.)}$$

Now if a_n be made infinite, the convergent reduces to p_{n-1}/q_{n-1}.

Hence $$\frac{p_{n-1}}{q_{n-1}}=\frac{A}{C},$$

and $$A=p_{n-1}, \quad C=q_{n-1}. \qquad \text{.....(ii.)}$$

Again, if a_n be made zero, the convergent reduces to p_{n-2}/q_{n-2}.

Hence $$\frac{p_{n-2}}{q_{n-2}}=\frac{B}{D},$$

and $$B=p_{n-2}, \quad D=q_{n-2}; \qquad \text{.....(iii.)}$$

therefore $$p_n=a_n p_{n-1}+p_{n-2},$$

$$q_n=a_n q_{n-1}+q_{n-2}.$$

PROF. STEGGALL.

27. *If $\frac{p_n}{q_n}$ be the n^{th} convergent of the continued fraction $\frac{a_1}{b_1+}\ \frac{a_2}{b_2+}\ \frac{a_3}{b_3+}\ldots$ it may always be assumed that $p_0=0$, $q_0=1$.*

In using the formulae $p_n=b_np_{n-1}+a_np_{n-2}$, $q_n=b_nq_{n-1}+a_nq_{n-2}$ for obtaining the successive convergents, it is always to be understood that if any convergent $\frac{p_r}{q_r}$ is not in its lowest terms, it must not be reduced. The same condition applies to $\frac{a_r}{b_r}$. For the first two convergents we have

$$p_1=a_1, \quad q_1=b_1, \quad p_2=a_1b_2, \quad q_2=b_1b_2+a_2.$$

Substituting these values in the formulae $p_2=b_2p_1+a_2p_0$, $q_2=b_2q_1+a_2q_0$, we get $p_0=0$, $q_0=1$.

The numerator of the above fraction is a_1, and the denominator is $b_1+\dfrac{a_2}{b_2+}\,\dfrac{a_3}{b_3+\ldots}$. Multiplying each of these by n, we see that the fraction may be written $\dfrac{na_1}{nb_1+}\,\dfrac{na_2}{b_2+}\,\dfrac{a_3}{b_3+\ldots}$. Thus the value of the continued fraction is not altered if a_r, b_r, a_{r+1} are multiplied by any the same number.

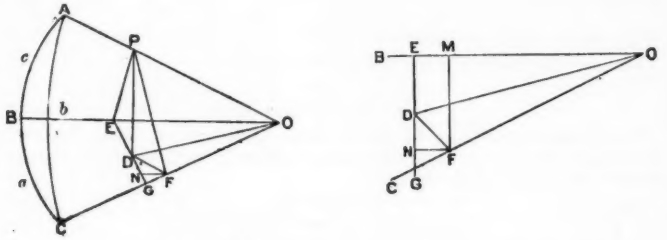
In this way the continued fraction $\dfrac{1}{a+}\,\dfrac{1}{b+}\,\dfrac{1}{a+}\,\dfrac{1}{b+}\,\dfrac{1}{a+\ldots}$ may be changed to the form $\dfrac{b}{ab+}\,\dfrac{ab}{ab+}\,\dfrac{ab}{ab+\ldots}$, which can easily be evaluated. EDITOR.

28. *Proofs of two formulae in Spherical Trigonometry.*

Let ABC be a spherical triangle, O the centre of the sphere.

Take any point P in OA, draw PD perpendicular to the plane BOC, and from D draw DE, DF perpendicular to OB, OC respectively; join PE, PF, OD.

Then it may be shown that PE is at right angles to OB, and PF at right angles to OC.



(1) Because $OE=OM+FN$,

$$\therefore\quad \frac{OE}{OP}=\frac{OM}{OF}\cdot\frac{OF}{OP}+\frac{FN}{DF}\cdot\frac{PF}{OP}\cdot\frac{DF}{PF};$$

or
$$\cos c=\cos a\cos b+\sin a\sin b\cos C.$$

(2) Because $EN=DE+DN$,

$$\therefore\quad \frac{EN}{OF}\cdot\frac{OF}{PF}=\frac{DE}{PD}\cdot\frac{PD}{PF}+\frac{DN}{DF}\cdot\frac{DF}{PF};$$

or
$$\sin a\cot b=\cot B\sin C+\cos a\cos C.$$

R. F. DAVIS.

29. *Proof that the medians and perpendiculars of a spherical triangle are concurrent.*

$$\frac{\sin P}{\sin\beta}=\frac{\sin\dfrac{a}{2}}{\sin PD},\quad \frac{\sin\gamma}{\sin P}=\frac{\sin AP}{\sin\dfrac{b}{2}},\quad \frac{\sin\beta}{\sin\gamma}=\frac{\sin\dfrac{b}{2}}{\sin a};$$

$\therefore$ multiplying, $\dfrac{\sin AP}{\sin PD}=2\cos\dfrac{a}{2}$; and, by symmetry, medians concur.

Concurrence of perpendiculars.

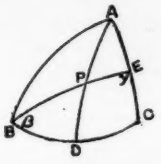
If, in same figure, AD, BE are perpendiculars,

$$\tan AD \cot B = \sin BD,$$

$$\sin BD = \cot\beta \tan PD,$$

$$\cot\beta \cot C = \cos a,$$

$$\therefore \quad \frac{\tan AD}{\tan PD} = \cos a \tan B \tan C;$$



and, by symmetry, the perpendiculars concur.

G. HEPPEL.

EXAMINATION QUESTIONS AND PROBLEMS.

Most of the following are from this year's Entrance Scholarship Papers at Cambridge and Oxford.

91. Sum the series

$$\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\frac{8}{1+x^8}+\ldots \text{ to } n \text{ terms.}$$

92. $ABCD$ is a quadrilateral inscribed in a circle. Show that the centres of the circles inscribed in the triangles ABC, ADC, ABD, CBD are at the corners of a rectangle.

93. Prove that the tangents at the extremities of two perpendicular chords of a circle form a quadrilateral which can be inscribed in a circle.

94. Given two straight lines which are not parallel but do not meet on the sheet of paper on which they are drawn, make a construction on the paper for the bisector of the angle between them.

95. Resolve into factors:

(i.) $(bcd+cda+dab+abc)^2-abcd(a+b+c+d)^2$;

(ii.) $\Sigma(a^3b^3+c^3d^3)(a-b)(c-d)$.

96. Solve the equation $ax^2+2bx+c+k(x^2+1)=0$.

Find between what limits of k it has equal roots, and show that no positive value of k will make the roots real unless the roots of the equation $ax^2+2bx+c=0$ are real.

97. If Q be the quotient and R the remainder when a multiple of 67 is divided by 10, prove that $Q-20R$ is also a multiple of 67.

98. Prove that if n^2 coins, of which exactly n are silver, are arranged at random in n rows, each containing n coins, the chance that one row at least occurs in which there is no silver coin is

$$1-\frac{(n-1)!\,(n^2-n)!\,n^{n-1}}{(n^2-1)!}.$$

99. Prove that $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}=\frac{1}{8}$.

100. Prove that if α, β, γ be any three angles connected by the relation

$$\cos\frac{\alpha+\beta+\gamma-\pi}{4}+\cos\frac{3\alpha-\beta-\gamma+\pi}{4}$$
$$+\cos\frac{3\beta-\gamma-\alpha+\pi}{4}+\cos\frac{3\gamma-\alpha-\beta+\pi}{4}=0,$$

then $$\tan\frac{\beta+\gamma-\alpha}{4}\tan\frac{\gamma+\alpha-\beta}{4}\tan\frac{\alpha+\beta-\gamma}{4}=1.$$

101. At a point within a rectangle the sides are found to subtend angles α, β, γ, δ taken in order. Prove that the ratio of the sides is given by the equation

$$x^2\sin\beta\sin\delta-x\sin(\beta+\delta)+\sin\alpha\sin\gamma=0.$$

102. Prove that the sum of the squares of the reciprocals of the lines joining one vertex of a regular polygon of n sides to the other $n-1$ vertices is $\frac{n^2-1}{12R^2}$, where R is the radius of the circumscribing circle.

103. Sum the series

$$\frac{\cos 2x}{\sin 3x}+\frac{\cos 6x}{\sin 9x}+\frac{\cos 18x}{\sin 27x}+\ldots \text{ to } n \text{ terms.}$$

104. Having given a directrix of a conic and the eccentricity and a point on the curve, show that the conic always touches a fixed conic having the same eccentricity.

105. Find the locus of the circumcentre of a triangle whose sides are the asymptotes and any tangent to a given hyperbola.

106. Chords of a conic are drawn through a fixed point, and through the pole of each chord a straight line is drawn parallel to the chord; show that the lines so drawn touch a parabola.

107. If two parabolas have a common focus, prove that they have only one real common finite tangent, and construct it.

108. Show that from any point $x'y'$ three normals real or imaginary can be drawn to the parabola $y^2=4ax$; and that when the normals are real the circle passing through their feet passes also through the vertex. Show that the equation to this circle is

$$x^2+y^2-\frac{y'y}{2}-(2a+x')x=0.$$

109. If P is any point on an ellipse whose semi-axes are a and b, S and S' the foci, and PS, PS' meet the ellipse in points Q and Q'; show that the envelope of QQ' is an ellipse whose semi-axes are a and $b^3/(2a^2-b^2)$.

110. Find the equation of the normal at the point $\theta=\alpha$ of the conic

$$l=r(1+e\cos\theta),$$

and prove that the part intercepted by the curve subtends at the origin an angle

$$2\tan^{-1}\frac{1+e^2+2e\cos\alpha}{e\sin\alpha}.$$

111. A triangle ABC, right-angled at C, formed of three uniform rods jointed at their ends, is suspended by a string attached to the middle point of AB. Show that the reaction at C is $cw/2\sqrt{2}$, where w is the weight per unit length of each rod.

112. Two equal smooth spheres of radius r and weight W are placed inside a hollow cylinder of radius a, open at both ends, which rests on a horizontal plane; prove that, in order that the cylinder may not upset, its weight must be at least

$$2W\left(1-\frac{r}{a}\right).$$

113. The coupling chain between an engine and a train, the mass of which is 96 tons, can bear a tension equal to the weight of 12 tons. Find the shortest time in which a speed of 30 miles an hour can be safely attained on a smooth level line.

114. A particle is projected from a fixed point above an inclined plane so as to strike the plane at right angles; show that the square of the least possible velocity of projection is

$$gc\{(\cos^2\alpha+4\sin^2\alpha)^{\frac{1}{2}}-\cos\alpha\},$$

where c is the perpendicular distance of the point from the plane, and α the inclination of the plane to the horizon.

115. Two equal smooth spheres of radius r move with the same speed in opposite directions in parallel lines at a distance c apart; prove that the motion of each deviates on impact through a right angle if

$$c^2(1+e)=4er^2,$$

where e is the coefficient of restitution.

116. Prove that if x and y be the rectangular coordinates of any point on any rectangular hyperbola having its centre at the origin

$$(x^2+y^2)\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^3-y\left(\frac{dy}{dx}\right)^2+x\frac{dy}{dx}-y=0.$$

117. Find the last 13 digits in the product $\underline{52}$.

R. P. ROYSTON.

118. Prove that any triangle is acute-angled or obtuse-angled according as the semi-perimeter is greater or less than $2R+r$.

R. F. DAVIS.

119. If the cubic equation $x+\frac{a^2}{x+b}+\frac{c^2}{x+d}+e=0$ has three real roots, one of the values of $\frac{a^2}{(x+b)^2}+\frac{c^2}{(x+d)^2}$ lies between 0 and 1, and the other two are both greater than 1; and if the cubic has only one real root, $\frac{a^2}{(x+b)^2}+\frac{c^2}{(x+d)^2}$ is greater than 1. EDITOR.

120. Find the prime factors of

5, 679, 431, 432, 056, 743, 205, 685, 679, 432.

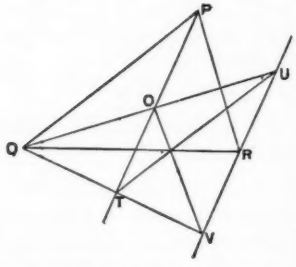
E. M. LANGLEY.

SOLUTIONS.

The following model solution of Problem 45, p. 19, has been received from A. E. STAMP:

On AB describe an equilateral triangle ABD; then the equilateral triangle CDE is the one required, having AC, AD, AE equal to CA, AB, BC. There are two solutions, since ABD may be described on either side of AB.

67. *Show how to solve a triangle when the lengths of the medians are given.*



Construct a triangle PQR whose sides are equal in length to the given medians, and let O be its centre of gravity. Produce PO, QO, QT to double their lengths to T, U, V respectively, then QVU is the triangle required.

It is easily seen that VU passes through and is bisected at R; and that VO, UT are equal and parallel to RP, PQ respectively.

To solve the triangle trigonometrically, suppose PQR the triangle required, so that OP, OQ, OR are given. Then the sides of the triangle OQT are given; and we can find QOT and POQ. Then in the triangle OPQ; OP, OQ, and POQ are known. Hence PQ can be found, and similarly PR, RQ.

69. *A ladder AB rests at an inclination of $\tan^{-1}\frac{1}{2}$ against the ground at A and a vertical wall at B, the coefficients of friction at A and B being $\frac{2}{3}$ and $\frac{1}{2}$. If the centre of gravity of the ladder is 6 feet from A and 9 feet from B, find how far a boy, whose weight is $\frac{1}{5}$ that of the ladder, can ascend before it begins to slip.*

Let W be the weight of the ladder, and $\frac{1}{5}W$ that of the boy; suppose that the ladder is in limiting equilibrium when the boy has climbed to a point x feet from B; let R be the vertical reaction of the ground at A, $\frac{2}{3}R$ the friction at A, R' the horizontal reaction of the wall at B, and $\frac{1}{2}R'$ the friction at B vertically upwards. Then resolving horizontally and vertically, we have

$$R'=\tfrac{2}{3}R, \quad R+\tfrac{1}{2}R'=\tfrac{6}{5}W, \qquad \therefore \quad R=\tfrac{9}{10}W.$$

Taking moments round B, $\quad 9W+\dfrac{x}{5}W+\tfrac{2}{3}R\times\tfrac{15}{2}-15R=0.$

This gives $x=0$; hence the ladder is in limiting equilibrium when the boy has just reached the top of it.

A geometrical solution by G. Heppel, showing that the resultant actions at A, B meet in the vertical line through the middle point of AB, has been received; but there is unfortunately not room to insert the solution in full.

70. *Solve the equations* $x^2+y^2=z^2-12$, $zx+y=13$, $yz-x=13$.

Squaring and adding the second and third equations $(z^2+1)(x^2+y^2)=338$; whence $(z^2+1)(z^2-12)=338$, $z^2=25$ or -14. The real solutions are

$$x=2, \ y=3, \ z=5, \ \text{and} \ x=-3, \ y=-2, \ z=-5.$$

71. *Out of a pack of 52 cards two have been lost. Three cards are drawn from the remaining 50 and found to be all aces; find the chance that a fourth drawing may also give an ace.*

When three cards have been drawn from the imperfect pack, it is known that the two missing cards do not include any one of these three, but are equally likely to be any two of the remaining 49 cards of the complete pack. Hence the chance of the next card drawn being an ace is the same as if the pack had been complete, *i.e.* the chance is $\frac{1}{13}$.

Solution also received from G. Heppel, giving $\frac{1}{55}$ as the answer.

73. *A hoop in the form of a parabola is held fixed in a vertical plane with its axis horizontal. A perfectly elastic particle is dropped from the upper end of the latus rectum; find the time and place of its second impact with the hoop.*

The path after impact is evidently a parabola, with vertex at the lower end of the latus rectum, and directrix a horizontal line through the upper. If the first parabola be $y^2=4ax$, the second will be $(x-a)^2+16a(y+2a)=0$; and the second point of impact is $x=9a$, $y=-6a$. The time between the two impacts is the time of falling vertically a distance $4a$ from rest, *i.e.* $2\sqrt{2a/g}$ seconds. G. Heppel.

74. *The equation $3x^4-8x^3+3x^2-2x-2=0$ is known to have two roots whose product is -1; find all the roots.*

We may assume that the given equation is identical with

$$(x^2-ax-1)(3x^2-\beta x+2)=0.$$

Multiplying out and identifying coefficients $a=2$, $\beta=2$. Hence the roots are

$$1\pm\sqrt{2}, \ \tfrac{1}{3}(1\pm\sqrt{-5}).$$

76. *A hyperbola cuts a concentric ellipse orthogonally at four points, not being a confocal. Prove that the tangents to the ellipse at adjacent points of intersection are perpendicular to one another.*

Let Q, Q' be adjacent points of intersection, the tangents and normals at which to the ellipse intersect in T, U respectively. Then CT bisects QQ'. Also since QU, $Q'U$ are tangents at Q, Q' to the hyperbola, CU bisects QQ'. Hence CU and CT must coincide, and TU bisects QQ'. But TU, the diameter of the circle $TQUQ'$, is not perpendicular to QQ', for the ellipse and hyperbola would in that case be confocal. Hence QQ' is a diameter of the circle $TQUQ'$, and the angle QTQ' is a right angle.

77. *Prove that, if*

$$\frac{b-c}{y-z}+\frac{c-a}{z-x}+\frac{a-b}{x-y}=0,$$

then

$$(b-c)(y-z)^2+(c-a)(z-x)^2+(a-b)(x-y)^2=0.$$

The determinant

$$\begin{vmatrix} b-c, & c-a, & a-b \\ x-y, & y-z, & z-x \\ z-x, & x-y, & y-z \end{vmatrix}$$

vanishes since the sum of the elements in each row vanishes. Consequently

$$(b-c)\{(y-z)^2-(z-x)(x-y)\}+\ldots+\ldots=0.$$

But, by hypothesis,

$$(b-c)(z-x)(x-y)+\ldots+\ldots=0;$$

hence also

$$(b-c)(y-z)^2+\ldots+\ldots=0.$$

B. TCHAYKOVSKY.

79. *Solve the equations*

$$u(2a-x)=x(2a-y)=y(2a-z)=z(2a-u)=b^2,$$

and prove that unless $b^2=2a^2$, $x=y=z=u$; *but that if* $b^2=2a^2$, *the equations are not independent.*

From the given equations we have

$$u=\frac{b^2}{2a-x}=\frac{b^2}{2a-}\,\frac{b^2}{2a-y}=\frac{b^2}{2a-}\,\frac{b^2}{2a-}\,\frac{b^2}{2a-}\,\frac{b^2}{2a-}\,\frac{u}{1}.$$

If $\frac{p_n}{q_n}$ is the n^{th} convergent to $\frac{b^2}{2a-}\,\frac{b^2}{2a-}\ldots$, we have

$$\frac{p_n}{q_n}=\frac{b^2}{2a-}\,\frac{p_{n-1}}{q_{n-1}};\qquad \therefore\ p_n=b^2q_{n-1};$$

$$\therefore\ u=\frac{p_4-up_3}{q_4-uq_3}=\frac{b^2(p_4-up_3)}{p_5-up_4}=\frac{b^2(p_4-up_3)}{(2a-u)p_4-b^2p_3};$$

$$\therefore\ p_4(2au-u^2-b^2)=0;$$

$$\therefore\ p_4=0,\ \text{or}\ u(2a-u)=b^2.$$

Now $p_4=4ab^2(2a^2-b^2)$; and it will be found that if either a or $2a^2-b^2$ vanishes, the equations are not independent; one of the four unknowns u, x, y, z can be chosen arbitrarily, and the others expressed in terms of it. If p_4 is not zero, we have $u(2a-u)=b^2$, and $u=x=y=z=a\pm\sqrt{a^2-b^2}$.

84. *Point out the fallacy in the following: If* x, y *be equal, and not zero, then* $x=y=\frac{x^2}{x}=\frac{y^2}{y}=\frac{x^2-y^2}{x-y}=x+y$, *which is absurd.* G. H. WARD.

The fallacy arises from the fact that although $\frac{x^2-y^2}{x-y}$ is equal to $x+y$, so long as x, y are unequal, no matter how small their difference; yet, when x, y are

absolutely equal, the value of $\frac{x^2-y^2}{x-y}$ cannot be assumed equal to $x+y$, but becomes indeterminate. The example shows that so-called *limiting values* are correct only when the limit is indefinitely approached, and not when the limit is actually reached. Thus $\frac{x^n-a^n}{x-a}$ has the value na^{n-1} when x is indefinitely near to a, but not when x is absolutely equal to a; the value of $\frac{x^n-a^n}{x-a}$ is indeterminate when $x-a$ is absolute zero.

85. *If $a-b$, $a-c$, $a-d$ are in H.P., then so also are $b-c$, $b-d$, $b-a$, and $c-d$, $c-a$, $c-b$, and $d-a$, $d-b$, $d-c$.* PROF. STEGGALL.

It will be found that the given condition reduces to

$$(a+c)(b+d)=2(ac+bd).$$

Therefore exchange a with c, or $\begin{pmatrix} a \text{ with } b \\ c \text{ with } d \end{pmatrix}$, or $\begin{pmatrix} a \text{ with } d \\ b \text{ with } c \end{pmatrix}$.

86. *What is the locus represented by the equation*

$$\{y^2+(a-x)^2\}\{x^2+(b-y)^2\}=\{ax+by-(x^2+y^2)\}^2?$$

Account geometrically for the result. E. M. LANGLEY.

This reduces on simplification to

$$(bx+ay-ab)^2=0.$$

The equation, in its original form, expresses the following geometrical property: If P be any point on the hypotenuse AB of a right-angled triangle ABC, and PM, PN be perpendiculars to CA, CB, then

$$AP.PB=CM.MA+CN.NB;$$

by putting $CA=a$, $CB=b$, $CM=x$, $CN=y$. E. M. LANGLEY.

Solution also received from G. HEPPEL.

88. *Prove that the difference of the cubes of two consecutive integers is a number terminating in one of the digits 1, 7, 9. Is it possible for this difference to be a perfect square?* R. F. DAVIS.

The difference can only have for its last digit one of those given by 1^3-0^3, 2^3-1^3, 3^3-2^3, ... 9^3-8^3, 10^3-9^3; which give 1, 7, 9, 7, 1, 1, 7, 9, 7, 1. Hence the last digit must be 1, 7, or 9.

The second part of the question can be answered in the affirmative, the number of solutions being infinite. It is required to find integral values of a, n which will satisfy the indeterminate equation

$$(n+1)^3-n^3=a^2, \text{ or } 4a^2-3(2n+1)^2=1.$$

Now, in the expansion of $(2+\sqrt{3})^{2r+1}$, it is clear that all the odd terms are even integers, and the even terms integral multiples of $\sqrt{3}$. Then putting

$$(2+\sqrt{3})^{2r+1}=2a+(2n+1)\sqrt{3}, \text{ and } (2-\sqrt{3})^{2r+1}=2a-(2n+1)\sqrt{3},$$

we have $$4a^2-3(2n+1)^2=(2+\sqrt{3})^{2r+1}(2-\sqrt{3})^{2r+1}=1.$$

Hence a general solution of the question is given by

$$2\sqrt{3}(2n+1)=(2+\sqrt{3})^{2r+1}-(2-\sqrt{3})^{2r+1},$$

where r has any integral value. If we denote this value of n by n_r, we have

$$n_{r+1}=14n_r-n_{r-1}+6.$$

But $n_0=0$, $n_1=7$; hence we get in succession $n_2=104$, $n_3=1455$, ... Thus, corresponding to the value of n_3, we have $1456^3-1455^3=2521^2$.

A similar question is suggested by considering even powers of $(2+\sqrt{3})$.

Putting $(2+\sqrt{3})^{2r}=(2a+1)+2n\sqrt{3}$, $(2-\sqrt{3})^{2r}=(2a+1)-2n\sqrt{3}$,

then $(2a+1)^2-12n^2=1$, or $(2n+1)^3-(2n-1)^3=2(2a+1)^2$.

R. P. ROYSTON.

Solution also received from G. HEPPEL.

89. *If c and d are the positive and negative roots of the equation*

$$(x+1)(x-\sqrt{5})=1,$$

prove that $(c+d\sqrt{3})\,cosec\ 1^\circ-(c\sqrt{3}-d)\,sec\ 1^\circ=8\sqrt{2}$. EDITOR.

Since $\sqrt{5}-1=4\sin 18^\circ=4\sin(45^\circ-27^\circ)=2\sqrt{2}(\cos 27^\circ-\sin 27^\circ)$,

and $\sqrt{5}+1=4\sin 54^\circ=8\sin 27^\circ.\cos 27^\circ$,

the equation for x may be written

$$x^2-2\sqrt{2}(\cos 27^\circ-\sin 27^\circ)x-8\sin 27^\circ.\cos 27^\circ=0;$$

the roots of which are $c=2\sqrt{2}\cos 27^\circ$, $d=-2\sqrt{2}\sin 27^\circ$.

Hence $c+d\sqrt{3}=4\sqrt{2}\left(\frac{1}{2}\cos 27^\circ-\frac{\sqrt{3}}{2}\sin 27^\circ\right)=4\sqrt{2}\sin(30^\circ-27^\circ)=4\sqrt{2}\sin 3^\circ$,

and $c\sqrt{3}-d=4\sqrt{2}\left(\frac{\sqrt{3}}{2}\cos 27^\circ+\frac{1}{2}\sin 27^\circ\right)=4\sqrt{2}\cos(30^\circ-27^\circ)=4\sqrt{2}\cos 3^\circ$;

therefore $(c+d\sqrt{3})\operatorname{cosec} 1^\circ-(c\sqrt{3}-d)\sec 1^\circ$

$$=4\sqrt{2}\left(\frac{\sin 3^\circ}{\sin 1^\circ}-\frac{\cos 3^\circ}{\cos 1^\circ}\right)=4\sqrt{2}\left(\frac{\sin 2^\circ}{\sin 1^\circ.\cos 1^\circ}\right)=8\sqrt{2}.$$

Solution also received from G. HEPPEL.

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We greatly regret to record the death of Professor Aurelio Lugli, Editor of the *Periodico di Matematica*, at Rome, on May 27th.

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